No. 1
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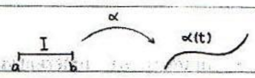
Curves

Def. A parametrized curve is a map $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ of an open interval $I = (a, b)$ into $\mathbb{R}^3$.

$\alpha(t) = (x(t), y(t), z(t))$. $I = (a, b)$ can as the whole line (i.e. $a \to -\infty$, $b \to \infty$)

Image of $\alpha \equiv C \equiv \{ (x(t), y(t), z(t)) \mid t \in I \}$ is called the trace of the curve



Def. A parametrized differential curve is a differentiable map $\alpha : I \to \mathbb{R}^3$

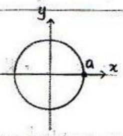
α is differentiable $(C^\infty) \Leftrightarrow x, y, z$ are C^∞

In general, $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector (velocity vector) of the curve at t.

Ex : (1) $\alpha : (-\varepsilon, 2\pi + \varepsilon) \to \mathbb{R}^2$, $\alpha(t) = (a\cos t, a\sin t)$, $a > 0$, $\alpha'(t) = (-a\sin t, a\cos t)$

$\beta(t) = (a\cos 2t, a\sin 2t)$, $\beta'(t) = (-2a\sin 2t, 2a\cos 2t)$ $\longrightarrow$ sint and cost are diff

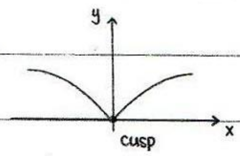
$|\beta'(t)| = 2|\alpha'(t)|$, α and β have same trace



(2) $\alpha(t) = (t^3, t^2)$, $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha'(t) = (3t^2, 2t)$ parametrized diff. curve

$\alpha(t)$ is the graph of $y = x^{\frac{2}{3}}$.

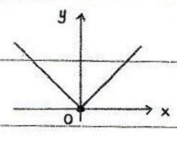
α is not a regular curve, $\alpha'(t)|_{t=0} = (0, 0)$



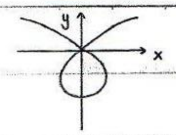
$\alpha'(0) = (0, 0)$

(3) $\alpha : I \to \mathbb{R}^2$, $\alpha(t) = (t, |t|)$, $y'(t) = \begin{cases} 1, & t \to 0^+ \\ -1, & t \to 0^- \end{cases}$

parametrized curve, but not parametrized differential curve



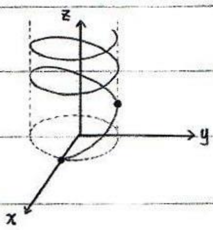
Note. A curve describes the motion of a particle in $\mathbb{R}^3$ and trace is the trajectory, but with different speed or director, the curve is consided to the different.



Ex : $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (t^3 - t, t^2 - 1)$, $\alpha(t)|_{t=1} = (0,0)$, $\alpha(t)|_{t=-1} = (0,0)$

α is not required one to one. * f is 1-1 $\Leftrightarrow$ $x \neq y \Rightarrow f(x) \neq f(y)$

Def. A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is regular, if $\alpha'(t) \neq 0$, $\forall t \in I$

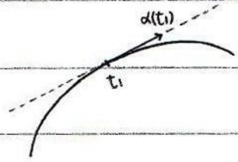


Ex : (helix) $\alpha : \mathbb{R} \to \mathbb{R}^3$ is given by $\alpha(t) = (a\cos t, a\sin t, bt)$, $a \neq 0$, $b \neq 0$

$\alpha(0) = (a, 0, 0)$, $\alpha(\frac{\pi}{2}) = (0, a, \frac{\pi b}{2})$, $\alpha(\pi) = (-a, 0, \pi b)$

$\alpha(t)$ has its trace in $\mathbb{R}^3$ a helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$

$\because$ $\alpha(t)$ is C^∞, $\alpha'(t) = (-a\sin t, a\cos t, b)$, $\alpha'(t) \neq 0$ $\forall t \in \mathbb{R}$ $\therefore$ $\alpha(t)$ is a regular curve.



Note. (1) Any point t st $\alpha'(t) = 0$ is a singular point of α.

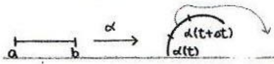
(2) let $\alpha : I \to \mathbb{R}^3$ be parametrized differentiable curve.

For each $t \in I$ st. $\alpha'(t) \neq 0$, the tangent line to α at t is the line which contains the point $\alpha(t)$ and the vector $\alpha'(t)$.

The length of the curve

Recall : If $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$, its length is $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Given $\alpha : [a,b] \to \mathbb{R}^3$, $\ell(\alpha[a,b]) \equiv$ the length of curve between a & b



the part of image of α between $\alpha(t)$ and $\alpha(t+\Delta t)$ is nearly a straight line

$\ell(\alpha[t+\Delta t]) = |\alpha(t+\Delta t) - \alpha(t)|$, $\Delta t \to 0$, $\frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} \sim \alpha'(t^*)$, $t^* \in (t, t+\Delta t)$

Take a partition, $P = \{ a = t_0 < t_1 < \cdots < t_n = b \}$, $\ell_a^b(\alpha, P) = \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|$, $|t_i - t_{i-1}|$ is small

Hw 1.3 (8) $|P| = \max_{1 \le i \le n} |t_i - t_{i-1}|$, given $\varepsilon > 0$, $\exists \delta > 0$ st. $\left| \int_a^b |\alpha'(t)| dt - \ell_a^b(\alpha, P) \right| < \varepsilon$, if $|P| < \delta$.

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Def. The arc-length of a regular curve $\alpha : I \to \mathbb{R}^3$ to a point $t_0 \in I$ is defined by $S(t) = \int_{t_0}^{t} |\alpha'(s)|\, ds$

(1) The arc-length $S(t)$ is a differentiable function.

(2) $\frac{ds}{dt} = |\alpha'(t)|$ by Fundamental Theorem of Calculus

(3) It depends on the image of α, not α

Ex: $\alpha(t) = (a\cos t, a\sin t)$, $a>0$, $t \in [0, 2\pi] \Rightarrow \alpha'(t) = (-a\sin t, a\cos t)$, $|\alpha'(t)| = a$, $S = \int_0^{2\pi} |\alpha'(t)|\, dt = 2\pi a$

Def. A regular curve $\alpha(s)$ is parametrized by arc-length if $|\alpha'(s)| = 1$

Fact: Every regular curve can be reparametrized by arc-length

Idea: Find $t(s)$ st. $\alpha(s)$ satisfies $|\alpha'(t(s))| = 1$

$\left|\frac{d\alpha(t(s))}{ds}\right| = \left|\frac{d\alpha}{dt}\frac{dt}{ds}\right| = 1 \Rightarrow \left|\frac{d\alpha}{dt}\right|\frac{dt}{ds} = 1 \Rightarrow \int\left|\frac{d\alpha}{dt}\right| dt = \int \frac{ds}{dt}\, dt = S(t)$ (may assume $\frac{dt}{ds} \geq 0$)

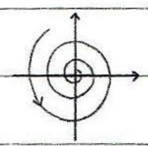
$\because \alpha$ is regular $\therefore S(t)$ is well-defined, $t(s)$ exists

Ex: (1) straight line $\alpha(t) = \vec{a}t + \vec{b}$, where $\vec{a}, \vec{b}$ are nonzero constant vectors.

$\alpha'(t) = \vec{a}$, $S(t) = \int_0^t |\alpha'(u)|\, du = |\vec{a}|t$ $\therefore t(s) = \frac{S}{|\vec{a}|}$

α can be reparametrized by arc-length, $\alpha(s) = \frac{\vec{a}}{|\vec{a}|}S + \vec{b}$

(2) $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $a>0$, $b<0$ logrithmic spiral



check α is regular : $\alpha'(t) = (ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t))$

$S = \int_0^t |\alpha'(u)|\, du = \int_0^t \sqrt{a^2e^{2bu}(b^2+1)}\, du = \int_0^t ae^{bu}\sqrt{b^2+1}\, du$ $\therefore S = \frac{a\sqrt{b^2+1}}{b}(e^{bt}-1)$

$\therefore t(s) = \frac{1}{b}\log\left(1 + \frac{bs}{a\sqrt{b^2+1}}\right)$ check : $\alpha(s) = \alpha(t(s))$, $|\alpha'(s)| = 1$

Let $\alpha(s)$ be a curve parametrized by arc-length. ($|\alpha'(s)| = 1$)

$\frac{d}{ds}(\alpha(s)) = \alpha'(s) = \vec{t}$ (tangent vector with $|\vec{t}| = 1$). $(\vec{t})' = \alpha''(s)$ measure the change of the tangent vector at $|\vec{t}|^2 = 1$

$|\vec{t}|^2 = 1$, $2\langle \vec{t}'$, $\vec{t}' \perp \vec{t}$, $\vec{t}' = k\vec{n}$, where $\vec{n}$ is the normal vector to tangent vector $\vec{t}$ and parallel to $(\vec{t})'$

Def. $k(s) = |\alpha''(s)|$ is called the curvature of the curve α.

A plane is determined by $\vec{t}(s)$ and $\vec{n}(s)$ is called osculating plane at s.

Ex : (1) straight line $\alpha(s) = \vec{a}s + \vec{b}$, $\alpha'(s) = \vec{a}$, $\alpha''(s) = 0$, $|\alpha'(s)| = |\vec{a}| = 1$, $k(s) \equiv 0$

(2) $\alpha(t) = (a\cos t, a\sin t)$, $\alpha'(t) = (-a\sin t, a\cos t)$, $|\alpha'(t)| = a$, $s = \int_0^t |\alpha'(u)|\,du = at$ $\therefore t = \frac{s}{a}$

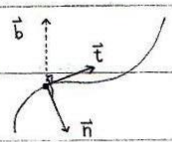
$\alpha(s) = \left(a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)\right)$, check $|\alpha'(s)| = 1$

$\alpha'(s) = \left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)\right) = \vec{t}$, $\alpha''(s) = (\vec{t})' = \left(-\frac{1}{a}\cos\left(\frac{s}{a}\right), -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right)$. $|\vec{t}| = |\alpha''(s)| = k(s) = \frac{1}{a}$, $\vec{n} = \frac{\vec{t}'}{k}$

Def. Let $\alpha(s)$ be a curve parametrized by arc-length st. $k(s) > 0$, $\forall s \in I$

$\vec{b} = \vec{t} \times \vec{n}$ is called binomial vector to α at s.

$\{\vec{t}, \vec{n}, \vec{b}\}$ is called Frenet frame (trihedrom).



(1) $|\vec{b}| = |\vec{t} \times \vec{n}| = |\vec{t}||\vec{n}|\sin\frac{\pi}{2} = 1$ $\because \vec{t} \perp \vec{n}$

(2) $\vec{b}$ is perpendicular to osculating plane (spanning by $\vec{t}$ and $\vec{n}$)

(3) $\vec{b}'$ measures how the osculating plane is moving

(4) $\vec{b}' = (\vec{t} \times \vec{n})' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}' = k\vec{n} \times \vec{n} + \vec{t} \times \vec{n}' = \vec{t} \times \vec{n}'$ $\therefore \vec{b}' \perp \vec{t}$

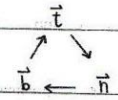
(5) $|\vec{b}|^2 = 1$, $\langle \vec{b}, \vec{b} \rangle = 1$, $2\langle \vec{b}', \vec{b} \rangle = 0$ $\therefore \vec{b}' \perp \vec{b}$

(6) $\vec{b}'$ is parallel to $\vec{n}$ $\therefore \vec{b}' = \square\vec{n}$

Def. Let $\alpha(s)$ be a curve parametrized by arc-length st. $\alpha'(s) \neq 0$, $\forall s \in I$

The number $\tau(s)$ defined by $\vec{b}' = \tau(s)\vec{n}(s)$ is called the torsion of α at s.

$$\begin{cases} \vec{t}' = k\vec{n}, \ k: \text{curvature} \\ \vec{b}' = \tau\vec{n}, \ \tau: \text{torsion} \end{cases} \Rightarrow \begin{cases} \langle \vec{n}, \vec{t} \rangle = 0 \\ \langle \vec{n}, \vec{b} \rangle = 0 \end{cases} \Rightarrow \begin{cases} \langle \vec{n}', \vec{t} \rangle = -\langle \vec{n}, \vec{t}' \rangle = -k \\ \langle \vec{n}', \vec{b} \rangle = -\langle \vec{n}, \vec{b}' \rangle = -\tau \end{cases}$$



$\vec{n} = \vec{b} \times \vec{t} \Rightarrow (\vec{n}') = (\vec{b}')' \times \vec{t} + \vec{b} \times \vec{t}' = \tau(s)\vec{n} \times \vec{t} + \vec{b} \times (k(s)\vec{n}) = -\tau(s)\vec{b} - k(s)\vec{t}$

$$\begin{cases} \vec{t}' = \qquad k\vec{n} \\ \vec{n}' = -k\vec{t} \qquad -\tau\vec{b} \\ \vec{b}' = \qquad \tau\vec{n} \end{cases}$$ is called the Frenet formula

Def. A plane spanned by $\vec{n}$ and $\vec{b}$ is called normal plane.

A plane spanned by $\vec{b}$ and $\vec{t}$ is called rectifying plane.

Ex : $\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right)$, where $c = \sqrt{a^2+b^2}$

$\vec{t} = \alpha'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$, $\vec{t}' = \alpha''(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$, $k(s) = |\alpha''(s)| = \frac{a}{c^2}$

$\vec{n} = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$, $\vec{b} = \vec{t} \times \vec{n} = \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)$

$\because \vec{b}' = \tau(s)\vec{n}$ $\quad \therefore \vec{b}' = \left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$, $\tau(s) = -\frac{b}{c^2}$

Thm. A curve in $\mathbb{R}^3$ which has $k > 0$ is a plane curve $\Leftrightarrow$ $\tau = 0$

<pf>($\Rightarrow$) Let $\alpha : I \to \mathbb{R}^3$ is a plane curve. Given any constant vector $\vec{\beta}$, constant r st. $\alpha(s) \cdot \vec{\beta} = r$

$(\alpha(s) \cdot \vec{\beta})' = (r)' = 0 \Rightarrow \alpha'(s) \cdot \vec{\beta} + \alpha(s) \cdot \overset{0}{\vec{\beta}'} = 0 \quad \therefore \alpha'(s) \cdot \vec{\beta} = 0$, $\vec{t} \cdot \vec{\beta} = 0 \Rightarrow \vec{t} \perp \vec{\beta}$

$(\alpha'(s) \cdot \vec{\beta})' = 0 \quad \therefore \alpha''(s)\vec{\beta} + \alpha'(s)\vec{\beta}' = 0 \quad \therefore \vec{n} \cdot \vec{\beta} = 0 \Rightarrow \vec{n} \perp \vec{\beta}$

$\Rightarrow \vec{b} = c\vec{\beta} \Rightarrow \vec{b}' = 0 \Rightarrow \tau = 0$

($\Leftarrow$) $\because \tau = 0 \quad \therefore \vec{b}' = 0 \Rightarrow \vec{b} \equiv$ constant vector

$(\alpha(s) \cdot \vec{b})' = \alpha'(s) \cdot \vec{b} + \alpha(s) \cdot \vec{b}' = \vec{t} \cdot \vec{b} = 0 \Rightarrow \alpha(s) \cdot \vec{b} = \text{const}$. $\therefore \alpha(s)$ is a plane curve.

No. 6

Date 106 9 21

Hw: Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc-length $k=0 \Leftrightarrow \alpha$ is a piece of line

Let $\alpha : I \to \mathbb{R}^3$ be a curve P by arc-length and $\alpha(s)$ has $k>0$ and $\tau = 0 \Rightarrow \alpha(s)$ is a part of circle of radius $\frac{1}{k}$

Find curvature k and torsion τ without assuming α is parametrized by arc-length α is a regular curve.

Since α can be reparametrized by arc-length, $\frac{d\alpha}{ds} = \frac{d\alpha}{dt}\frac{dt}{ds} \Rightarrow 1 = |\frac{d\alpha}{ds}| = |\frac{d\alpha}{dt}|\frac{dt}{ds}$ $\therefore \frac{dt}{ds} = \frac{1}{|\alpha'|} \cdots (1)$

$\vec{t} = \alpha' \frac{dt}{ds} = \frac{\alpha'}{|\alpha'|}$ $(\vec{t})' = \frac{d}{ds}(\alpha' \frac{dt}{ds}) = \frac{d}{ds}(\alpha')\frac{dt}{ds}\frac{dt}{ds} + \alpha' \frac{d^2t}{ds^2} = \alpha''(\frac{dt}{ds})^2 + \alpha' \frac{d^2t}{ds^2} = \frac{\alpha''}{|\alpha'|^2} + \alpha' \frac{d^2t}{ds^2} \cdots (2)$

$(2)\cdot\alpha' \Rightarrow 0 = kn\cdot\alpha' = \frac{\alpha''\cdot\alpha'}{|\alpha'|^2} + |\alpha'|^2\frac{d^2t}{ds^2} \Rightarrow \frac{d^2t}{ds^2} = \frac{-\alpha''\cdot\alpha'}{|\alpha'|^4}$ $\therefore \vec{t}' = k\vec{n} = \frac{\alpha''}{|\alpha'|^2} + \alpha'\left(\frac{-\alpha''\cdot\alpha'}{|\alpha'|^4}\right) \cdots (3)$

$k = \sqrt{\left(\frac{\alpha''}{|\alpha'|^2}\right)^2 + \left(\frac{\alpha'(-\alpha''\cdot\alpha')}{|\alpha'|^4}\right)^2 - 2\frac{(\alpha''\cdot\alpha')^2}{|\alpha'|^6}} = \sqrt{\frac{|\alpha''|^2\cdot|\alpha'|^2}{|\alpha'|^4\cdot|\alpha'|^2} - \frac{(\alpha''\cdot\alpha')^2}{|\alpha'|^6}} \underline{\underline{(\alpha''\cdot\alpha')^2 = |\alpha''|^2|\alpha'|^2\cos^2\theta}} \sqrt{\frac{|\alpha''|^2\cdot|\alpha'|^2(1-\cos^2\theta)}{|\alpha'|^6}} = \sqrt{\frac{|\alpha''\times\alpha'|^2}{|\alpha'|^6}} = \frac{|\alpha''\times\alpha'|}{|\alpha'|^3} \cdots (*)$

$\vec{n} = \frac{|\alpha'|\alpha''}{|\alpha'\times\alpha''|} - \frac{(\alpha''\cdot\alpha')\cdot\alpha'}{|\alpha'||\alpha'\times\alpha''|} = \frac{|\alpha'|^2\alpha'' - (\alpha''\cdot\alpha')\alpha'}{|\alpha'||\alpha'\times\alpha''|}$ $\vec{b} = \vec{t}\times\vec{n} = \frac{\alpha'}{|\alpha'|}\times\left(\frac{|\alpha'|^2\alpha'' - (\alpha''\cdot\alpha')\alpha'}{|\alpha'||\alpha'\times\alpha''|}\right) = \frac{\alpha'\times\alpha''}{|\alpha'\times\alpha''|}$

$\vec{b}' = \frac{\alpha'\times\alpha''}{|\alpha'||\alpha'\times\alpha''|} - \frac{\alpha'\times\alpha''[(\alpha'\times\alpha'')\cdot(\alpha'\times\alpha'')]}{|\alpha'||\alpha'\times\alpha''|^3}\frac{dt}{ds}$

$\because \vec{b}' = \tau\vec{n}$ $\therefore \tau = \vec{b}'\cdot\vec{n} = \frac{(\alpha'\times\alpha''')\alpha''}{|\alpha'\times\alpha''|^2} \cdots (**)$

Let $\alpha(t) = (x(t), y(t))$, $\alpha : I \to \mathbb{R}^2 \Rightarrow \vec{t} = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{(x'(t), y'(t))}{\sqrt{(x')^2+(y')^2}}$, $\vec{t}' = \frac{(x'', y'')}{(x'^2+y'^2)} - \frac{x''\cdot x' + y''\cdot y'\cdot(x'\cdot y')}{(x'^2+y'^2)^2}$

$\because$ on $\mathbb{R}^2$ $\therefore \vec{t}\perp\vec{n}$, $\vec{t}' = k\vec{n} \Rightarrow \vec{n} = \frac{(y'(t), x'(t))}{\sqrt{(x')^2+(y')^2}}$, $k = \vec{t}'\cdot\vec{n} = \frac{-x''y' + y''x'}{(x'^2+y'^2)^{3/2}}$ on $\mathbb{R}^3$, $k>0$

positive determinant.

Def. A rigid motion in $\mathbb{R}^3$ is a result of composing a translation with an orthogonal transformation with

Recall: (1) A translation by a vector $v \in \mathbb{R}^3$ is a map $A : \mathbb{R}^3 \to \mathbb{R}^3$ that is given by $A(p) = p + v$, $p \in \mathbb{R}^3$

(2) A linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation if $\rho_u \cdot \rho_v = u \cdot v$, $\forall u, v \in \mathbb{R}^3$

Remark. arc-length, curvature and torsion are invariant under rigid motion

$M : \mathbb{R}^3 \to \mathbb{R}^3$ rigid motion, $\alpha(t)$ a curve, $\bar{\alpha}(t) = M\circ\alpha = \underset{\text{rotation}}{\underline{\underline{\rho}}}\circ\alpha + \underset{\text{transformation}}{\underline{\underline{C}}}$, $\int_a^b |\frac{d\alpha}{dt}|\,dt = \int_a^b |\frac{d(M\circ\alpha)}{dt}|\,dt$

Thm. Fundamental theorem of the curve (local)

Given differential function $k(s)>0$, $\tau(s)$, $s\in I$, $\exists$ a regular curve parametrized by arc-length $\alpha: I\to\mathbb{R}^3$

s is the arc-length of α, $k(s)$ is the curvature of α, $\tau(s)$ is the torsion of α.

Moreover, α is unique up to rigid motion, i.e. $\exists$ rigid motion $M:\mathbb{R}^3\to\mathbb{R}^3$, $\bar{\alpha}=M\circ\alpha=\rho\circ\alpha+c$

<pf> (uniqueness) Let $\alpha(s)$ and $\bar{\alpha}(s)$ be two curve parametrized by $s\in[a,b]$

$\exists$ s is their arc-length and $k(s)$, $\tau(s)$, $\bar{k}(s)$, $\bar{\tau}(s)$ satisfied $k(s)=\bar{k}(s)$, $\tau(s)=\bar{\tau}(s)$

Let $s_0\in[a,b]$, consider $\beta(s)=\bar{\alpha}(s)-(\bar{\alpha}(s_0)-\alpha(s_0))$, $\beta(s_0)=\bar{\alpha}(s_0)-(\bar{\alpha}(s_0)-\alpha(s_0))=\alpha(s_0)$

$\therefore \beta(s)\equiv$ a translation of $\alpha(s)$

Now rotate $\beta+\beta'(s_0)=\alpha'(s_0)$ call the resulting curve $r(s)$ rotate again to match the normal vector at s_0.

Now all the resulting curve $\theta(s)$ $\therefore \theta(s), \alpha(s)$ have identical k and τ and $\alpha(s_0)=\theta(s_0)$, $\alpha'(s_0)=\theta'(s_0)$, $\alpha''(s_0)=\theta''(s_0)$ and also with identical binomal vector at s_0.

Let $\{\vec{t},\vec{n},\vec{b}\}$ and $\{\vec{\bar{t}},\vec{\bar{n}},\vec{\bar{b}}\}$ be the Frenet frame for $\alpha(s)$ and $\theta(s)$

consider $f(s)=|t-\bar{t}|^2+|n-\bar{n}|^2+|b-\bar{b}|^2$ $(|\alpha|^2=\langle\alpha,\alpha\rangle)$

$f'(s)=2[\langle t-\bar{t}, t'-\bar{t}'\rangle+\langle n-\bar{n}, n'-\bar{n}'\rangle+\langle b-\bar{b}, b'-\bar{b}'\rangle]$

$=2[k\langle t-\bar{t}, n-\bar{n}\rangle-k\langle n-\bar{n}, t-\bar{t}\rangle+\tau\langle n-\bar{n}, -b+\bar{b}\rangle+\tau\langle b-\bar{b}, n-\bar{n}\rangle]=0 \Rightarrow f'(s)=0$ $\therefore f(s)$ const.

$\because f(s_0)=0 \Rightarrow f(s)\equiv 0$, $\forall s\in[a,b] \Rightarrow t=\bar{t}$, $n=\bar{n}$, $b=\bar{b}$

consider $g(s)=|\alpha(s)-\theta(s)|^2$, $g'(s)=2\langle\alpha(s)-\theta(s), \alpha'(s)-\theta'(s)\rangle=2\langle\alpha(s)-\theta(s), t-\bar{t}\rangle=0$, $\forall s\in[a,b]$

However, $g(s_0)=|\alpha(s_0)-\theta(s_0)|^2=0 \Rightarrow g(s)\equiv 0 \Rightarrow \alpha(s)=\theta(s)$

<pf> (existance of $\alpha(s)$) We want to construct curve from $k(s)>0$ and $\tau(s)$, $t(s)=\alpha'(s)$

construct $\{\vec{t},\vec{n},\vec{b}\}$ satisfy the Frenet formula

If we set $t=(t_1,t_2,t_3)$, $n=(n_1,n_2,n_3)$, $b=(b_1,b_2,b_3)$

$$* \begin{cases} t_1'=kn_1 & n_1'=-kt_1-\tau b_1 & b_1'=\tau n_1 \\ t_2'=kn_2 & n_2'=-kt_2-\tau b_2 & b_2'=\tau n_2 \\ t_3'=kn_3 & n_3'=-kt_3-\tau b_3 & b_3'=\tau n_3 \end{cases} \Rightarrow \begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & kI & 0 \\ -kI & 0 & -\tau I \\ 0 & \tau I & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

There are nine equation $*$ has unknown, $*$ is 9×9 linear ODE

By ODE Thm., if we prescribe the volumn of t, n, b at a fixed point, then $\exists$ sol. to the system.

We take $t(s_0) = (1, 0, 0)$, $n(s_0) = (0, 1, 0)$, $b(s_0) = (0, 0, 1)$

If $h : [a,b] \to \mathbb{R}^9$, $h(s) = (t(s), n(s), b(s))$, $h(s_0) = (e_1, e_2, e_3)$ $\therefore h'(s) = A(s)h(s)$, $h(s_0) = (100010001)$

By ODE Thm $\exists!$ sol. $g : [a,b] \to \mathbb{R}^9$ to the IVP. Claim: $\{t, n, b\}$ are orthonormal.

$(|t|^2)' = 2\langle t', t\rangle = 2k\langle t, n\rangle$ $\qquad$ $(\langle t, n\rangle)' = k|n|^2 - k|t|^2 - 2\langle t, b\rangle$

$(|n|^2)' = 2\langle n', n\rangle = -2k\langle n, t\rangle - 2\tau\langle n, b\rangle$ $\qquad$ $(\langle n, b\rangle)' = 2|n|^2 - k\langle b, t\rangle - \tau|b|^2$

$(|b|^2)' = 2\langle b', b\rangle = 2\tau\langle n, b\rangle$ $\qquad$ $(\langle t, b\rangle)' = k\langle n, b\rangle + \tau\langle t, n\rangle$

ℓ: vector value function, $\ell : [a,b] \to \mathbb{R}^6$; $\ell(s) = (|t|^2, |n|^2, |b|^2, \langle t, n\rangle, \langle n, b\rangle, \langle b, t\rangle)$

$$
⑥ \begin{cases} \dfrac{d\ell}{ds} = M(s)\,\ell(s) \\[2ex] \ell(s)\big|_{s=s_0} = (1, 1, 1, 0, 0, 0) \end{cases}, \text{ where } M(s) = \begin{pmatrix} & & & 2k(s) & 0 & 0 \\ & \bigcirc & & -2k(s) & -2\tau(s) & 0 \\ & & & 0 & 2\tau(s) & 0 \\ -k(s) & k(s) & 0 & 0 & 0 & -\tau(s) \\ 0 & \tau(s) & -\tau(s) & 0 & 0 & -k(s) \\ 0 & 0 & 0 & \tau(s) & k(s) & 0 \end{pmatrix}
$$

consider const curve $u(s) = (1, 1, 1, 0, 0, 0)$ is a sol of ⑥ ($\because$ ⑥ linear ODE)

By ODE uniqueness of sol $\Rightarrow u(s) = \ell(s) \Rightarrow \ell(s) = (1, 1, 1, 0, 0, 0)$

$$
\therefore \text{We have} \begin{cases} |t|^2 = 1 & \langle t, n\rangle = 0 \\ |n|^2 = 1 & \langle n, b\rangle = 0 \quad \forall s \in [a,b] \\ |b|^2 = 1 & \langle b, t\rangle = 0 \end{cases}
$$

$\{t, n, b\}$ satisfy the Frenet formula $\Rightarrow \alpha(s) = \int_{s_0}^{s} t(u)\,du = \int_{s_0}^{s} \alpha'(u)\,du$

Global property of plane curve

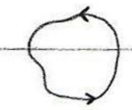
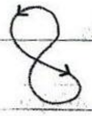
Def. A curve $\alpha : [a,b] \to \mathbb{R}^2$ is a closed curve, if $\alpha(a) = \alpha(b), \dots, \alpha^{(n)}(a) = \alpha^{(n)}(b)$, where α and its derivatives agree at a and b.

Def. A simple closed curve in $\mathbb{R}^3$ is a curve with no self-intersection,

i.e. $\alpha : [a,b] \to \mathbb{R}^3$ and $\alpha(t_1) = \alpha(t_2)$, for some $t_1, t_2 \in [a,b]$, then $t_1 = t_2$.

Def. A plane curve $\alpha : [a,b] \to \mathbb{R}^2$ is a positively oriented simple closed curve if the interior re

enclosed by curve is on the LHS.

Ex :  ✓  ×

Thm. (Isoperimetric Inequality) Let $\alpha : [a,b] \to \mathbb{R}^2$ be a positive oriented simple closed curve in $\mathbb{R}^2$

Denote ℓ = length of α , A = area of region enclosed (bounded) by α in $\mathbb{R}^2$

Then $4\pi A \le \ell^2$ equality holds $\Leftrightarrow$ the image of α is circle.

Recall : Green Theorem

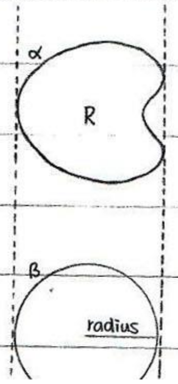
Let c be a positively oriented piecewise smooth simple closed curve in $\mathbb{R}^2$ and let R be the region enclosed by C.

If f and g are function (x, y) defined on open region containing R and have "continuous partial derivate", then $\int_C \left(f \frac{dx}{dt} + g \frac{dy}{dt} \right) dt = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy$ $\quad f, g \in C^1(\mathbb{R}^2)$

* Let $f = -y$, $g = x$, then $\int_C (-y x' + x y')\, dt = \iint_R 2\, dx\, dy = 2 \iint_R dx\, dy = 2\, \text{Area}(R)$

$\therefore$ Area(R) : $A = \frac{1}{2} \int_C (x y' - y x')\, dt \cdots (*)$

$A \le \frac{\ell^2}{4\pi}$



$A_c = \pi r^2$

diameter = $d = 2r$

Let curve α be parametried by arc-length $\alpha(s) = (x(s), y(s))$, $s \in [a$

Let curve β (circle) be parametried by arc-length $\beta(s) = (x(s), \bar{y}(s))$

$x^2(s) + \bar{y}^2(s) = r^2$

we know that $A(R)$ = area of region $R = \frac{1}{2}\int_0^\ell (x\acute{y} - y\acute{x})\, ds$

$\because \int_0^\ell (\acute{x}y)\, ds = (xy)\Big|_0^\ell - \int_0^\ell x\acute{y}\, ds \quad \therefore \int_0^\ell (\acute{x}y)\, ds = -\int_0^\ell x\acute{y}\, ds,\ \int_0^\ell x\acute{y}\, ds = -\int_0^\ell y\acute{x}\, ds$

$A(R) = \frac{1}{2}\int_0^\ell (x\acute{y} - y\acute{x})\, ds = \frac{1}{2}\int_0^\ell (x\acute{y} + x\acute{y})\, ds = \int_0^\ell x\acute{y}\, ds$

$A(Cir) = \pi r^2 = \frac{1}{2}\int_0^\ell (x\bar{y}' - \bar{y}\acute{x})\, ds = \frac{1}{2}\int_0^\ell (-\bar{y}\acute{x} - \bar{y}\acute{x})\, ds = -\int_0^\ell (\bar{y}\acute{x})\, ds$ $\qquad \acute{x}^2 + \acute{y}^2 = 1 \ \because$ by arc-length

$A(R) + A(Cir) = A(R) + \pi r^2 = \int_0^\ell (x\acute{y} - \bar{y}\acute{x})\, ds \le \int_0^\ell \sqrt{(x\acute{y} - \bar{y}\acute{x})^2}\, ds \le \int_0^\ell \sqrt{(x^2+\bar{y}^2)(\acute{x}^2+\acute{y}^2)}\, ds = \int_0^\ell r\, ds = \ell r \cdots (**)$

$\sqrt{A(R)}\sqrt{A(Cir)} \le \frac{1}{2}[A(R) + A(Cir)] = \frac{1}{2}[A(R) + \pi r^2] \le \frac{\ell r}{2} \Rightarrow A(R)\pi r^2 \le \frac{\ell^2 r^2}{4} \Rightarrow A(R) \le \frac{\ell^2}{4\pi}$

$A = \frac{\ell^2}{4\pi}$: $\int_0^\ell \sqrt{(xy' - \bar{y}x')^2}\, ds = \int_0^\ell \sqrt{(x^2+\bar{y}^2)(x'^2+y'^2)}\, ds \Rightarrow -2x\acute{y}\bar{y}x' = x^2x'^2 + y'^2\bar{y}^2$

$(x\acute{x} + \bar{y}\acute{y})^2 = 0 \Rightarrow x\acute{x} = -\bar{y}\acute{y},\ (x\acute{y} - \bar{y}x')^2 = x^2\acute{y}^2 + 2x^2x'^2 + \bar{y}^2x'^2 = x^2 + x'^2r^2 = r^2$

$\therefore x^2 = r^2(1 - x'^2) \Rightarrow x = \pm r\acute{y},\ y = \pm rx' \qquad \therefore x^2 + y^2 = r^2\acute{y}'^2 + r^2x'^2 = r^2$

Remark. Isoperimetric inequality also holds for α, where α is piecewise continuous allow finite # cornor.